

# 多维 Laplace 变换及其应用

范洪福

(华东工业大学基础部 上海 200093)

**摘 要** 提出了多维 Laplace 变换的概念, 并讨论它在解微分方程中的应用.

**关键词** Laplace 变换; 微分方程; 留数

**中图法分类号** O 177.6

Laplace 变换是一类重要的积分变换, 在线性系统和数理方程中是一种极其重要的工具, 给解微分方程带来极大方便. 但是, 迄今为止, 人们只限于讨论一维 Laplace 变换, 这极大地限制了它的应用, 有必要将 Laplace 变换的概念进行拓广. 本文提出多维 Laplace 变换的概念及其性质, 并着重讨论它在解微分方程中的应用.

## 1 多维 Laplace 变换及其逆变换

**定义** 设  $n$  元函数  $f(t_1, t_2, \dots, t_n)$  在  $t_1 \geq 0, t_2 \geq 0, \dots, t_n \geq 0$  上有定义, 且积分

$$\int_0^{+\infty} \int_0^{+\infty} \cdots \int_0^{+\infty} f(t_1, t_2, \dots, t_n) e^{-(s_1 t_1 + s_2 t_2 + \cdots + s_n t_n)} dt_1 dt_2 \cdots dt_n$$

在  $C^n$  的某一区域内收敛 ( $(s_1, s_2, \dots, s_n) \in C^n$ ,  $C^n$  是  $n$  维复数域), 则称此积分为  $f(t_1, \dots, t_n)$  的  $n$  维 Laplace 变换 (或称象函数), 记为

$$\begin{aligned} F(s_1, s_2, \dots, s_n) &= \mathcal{L}[f(t_1, t_2, \dots, t_n)] = \mathcal{L}[f] \\ &= \int_0^{+\infty} \cdots \int_0^{+\infty} f(t_1, \dots, t_n) e^{-(s_1 t_1 + \cdots + s_n t_n)} dt_1 dt_2 \cdots dt_n; \end{aligned}$$

称  $f(t_1, \dots, t_n)$  为  $F(s_1, \dots, s_n)$  的  $n$  维 Laplace 逆变换 (或称象原函数), 记为

$$f(t_1, \dots, t_n) = \mathcal{L}^{-1}[F(s_1, \dots, s_n)] = \mathcal{L}^{-1}[F].$$

**$n$  维 Laplace 变换的存在定理** 若  $f(t_1, \dots, t_n)$  满足

a. 在  $[0, +\infty)^n$  的任一有界区域上分段连续;

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b. 存在常数  $M \geq 0, c_1 \geq 0, c_2 \geq 0, \dots, c_n \geq 0$ , 使得  $|f(t_1, \dots, t_n)| \leq M \cdot e^{c_1 t_1 + c_2 t_2 + \dots + c_n t_n}, \forall (t_1, \dots, t_n) \in [0, +\infty)^n$ , (此时称  $f(t_1, \dots, t_n)$  的增长是指数级的,  $c_1, c_2, \dots, c_n$  为  $f$  的增长指数). 则  $f(t_1, \dots, t_n)$  的  $n$  维 Laplace 变换

$$F(s_1, \dots, s_n) = \int_0^{+\infty} \dots \int_0^{+\infty} f(t_1, \dots, t_n) e^{-(s_1 t_1 + \dots + s_n t_n)} dt_1 dt_2 \dots dt_n \quad (1)$$

在区域  $\operatorname{Re}(s_1) > c_1, \operatorname{Re}(s_2) > c_2, \dots, \operatorname{Re}(s_n) > c_n$  内存在, 且函数  $F(s_1, \dots, s_n)$  关于复变量  $s_1, \dots, s_n$  是  $n$  元全纯函数. 此时右端积分绝对收敛而且一致收敛.

证 设  $s_i = \beta_i + j\omega_i (i = 1, 2, \dots, n, j = \sqrt{-1})$ , 则当  $\beta_i - c_i \geq \varepsilon_i > 0$ , 即  $\beta_i \geq c_i + \varepsilon_i$  时, 有

$$\begin{aligned} |f(t_1, \dots, t_n) e^{-(s_1 t_1 + \dots + s_n t_n)}| &\leq M e^{c_1 t_1 + \dots + c_n t_n} \cdot e^{-[(c_1 + \varepsilon_1)t_1 + \dots + (c_n + \varepsilon_n)t_n]} \\ &= M e^{-(\varepsilon_1 t_1 + \dots + \varepsilon_n t_n)}, \end{aligned}$$

$$\begin{aligned} \text{故} \quad &\int_0^{+\infty} \dots \int_0^{+\infty} |f(t_1, \dots, t_n) e^{-(s_1 t_1 + \dots + s_n t_n)}| dt_1 dt_2 \dots dt_n \\ &\leq \int_0^{+\infty} \dots \int_0^{+\infty} M e^{-(\varepsilon_1 t_1 + \dots + \varepsilon_n t_n)} dt_1 dt_2 \dots dt_n = \frac{M}{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n}, \end{aligned}$$

因此式(1)的右端不仅绝对收敛而且一致收敛.

由于  $e^{-(s_1 t_1 + \dots + s_n t_n)}$  关于  $s_1, \dots, s_n$  的  $n$  重幂级数展开式在  $(t_1, \dots, t_n) \in R^n$  及  $(s_1, \dots, s_n) \in C^n$  内的任何有界区域上一致收敛和绝对收敛, 故由已证结果可知, 在区域  $\operatorname{Re}(s_1) > c_1, \dots, \operatorname{Re}(s_n) > c_n$  内  $F(s_1, \dots, s_n)$  是  $n$  元全纯函数. 证毕.

实践中, 许多常见的多元函数均满足存在定理的条件, 因此, 它们的多维 Laplace 变换是存在的. 下面举一例说明之.

**例1** 求函数  $f(t_1, t_2, t_3) = e^{at_1} + e^{bt_2} + e^{ct_3}$  的三维 Laplace 变换.

**解** 因  $|e^{at_1} + e^{bt_2} + e^{ct_3}| \leq 3e^{c_1 t_1 + c_2 t_2 + c_3 t_3}, (t_1, t_2, t_3 \geq 0)$ , 其中  $c_1 = \max\{a, 0\}, c_2 = \max\{b, 0\}, c_3 = \max\{c, 0\}$ , 所以,  $f(t_1, t_2, t_3)$  的三维 Laplace 变换存在, 且

$$\begin{aligned} F(s_1, s_2, s_3) &= \mathcal{L}[f(t_1, t_2, t_3)] = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} (e^{at_1} + e^{bt_2} + e^{ct_3}) \cdot e^{-(s_1 t_1 + s_2 t_2 + s_3 t_3)} dt_1 dt_2 dt_3 \\ &= \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} e^{-(s_1 - a)t_1} \cdot e^{-(s_2 t_2 + s_3 t_3)} dt_1 dt_2 dt_3 \\ &\quad + \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} e^{-(s_2 - b)t_2} \cdot e^{-(s_1 t_1 + s_3 t_3)} dt_1 dt_2 dt_3 \\ &\quad + \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} e^{-(s_3 - c)t_3} \cdot e^{-(s_1 t_1 + s_2 t_2)} dt_1 dt_2 dt_3 \\ &= \frac{1}{(s_1 - a)s_2 s_3} + \frac{1}{s_1 (s_2 - b)s_3} + \frac{1}{s_1 s_2 (s_3 - c)}, \quad (\operatorname{Re}(s_i) > c_i, \quad i = 1, 2, 3). \end{aligned}$$

关于多维 Laplace 逆变换,我们有如下的反演公式.

**定理** 设  $f(t_1, t_2, \dots, t_n)$  满足存在定理的条件,记

$$F_i(s_1, s_2, \dots, s_i) = \int_0^{+\infty} \cdots \int_0^{+\infty} f(t_1, \dots, t_n) e^{-(s_1 t_1 + s_2 t_2 + \cdots + s_i t_i)} dt_1 dt_2 \cdots dt_i,$$

( $i = 1, 2, \dots, n$ ). 若  $s_{i1}, s_{i2}, \dots, s_{ik_i}$  是  $F_i(s_1, \dots, s_i)$  关于复变量  $s_i$  的所有奇点,且  $\lim_{s_i \rightarrow \infty} F_i(s_1, \dots,$

$s_i) = 0$ ,  $\sigma_i > c_i$ ,  $i = 1, \dots, n$ , 则  $f(t_1, \dots, t_n) = \frac{1}{(2\pi j)^n} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} \cdots \int_{\sigma_n - j\infty}^{\sigma_n + j\infty} F(s_1, \dots, s_n) e^{s_1 t_1 + s_2 t_2 + \cdots + s_n t_n}$

$$ds_1 ds_2 \cdots ds_n = \sum_{l_1=1}^{k_1} \text{Res}_{s_1=s_{1l_1}} \left\{ \cdots \left\{ \sum_{l_n=1}^{k_n} \text{Res}_{s_n=s_{nl_n}} \left\{ \sum_{l_n=1}^{k_n} \text{Res}_{s_n=s_{nl_n}} [F(s_1, \dots, s_n) e^{s_n t_n}] \right\} \cdot e^{s_{n-1} t_{n-1}} \right\} \cdots \right\} e^{s_1 t_1}.$$

**证** 首先,函数  $F_i(s_1, \dots, s_i)$  实际上就是  $f(t_1, \dots, t_n)$  关于  $t_1, \dots, t_i$  的  $i$  维 Laplace 变换,因此,根据存在定理,  $F_i(s_1, \dots, s_i)$  是存在的,而且关于  $s_i$  在区域  $\text{Re}(s_i) > c_i$  内解析.

其次,由于

$$F_i(s_1, \dots, s_i) = \int_0^{+\infty} \left[ \int_0^{+\infty} \cdots \int_0^{+\infty} f(t_1, \dots, t_n) e^{-(s_1 t_1 + \cdots + s_i t_i)} dt_1 \cdots dt_{i-1} \right] \cdot e^{-s_i t_i} dt_i,$$

根据文献[1],有  $F_{i-1}(s_1, \dots, s_{i-1}) = \frac{1}{2\pi j} \int_{\sigma_i - j\infty}^{\sigma_i + j\infty} F_i(s_1, \dots, s_i) e^{s_i t_i} ds_i$

$$= \sum_{l_i=1}^{k_i} \text{Res}_{s_i=s_{il_i}} [F_i(s_1, \dots, s_i) e^{s_i t_i}], \quad (i = 1, 2, \dots, n).$$

故  $f(t_1, t_2, \dots, t_n) = \frac{1}{2\pi j} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F_1(s_1) e^{s_1 t_1} ds_1 = \sum_{l_1=1}^{k_1} \text{Res}_{s_1=s_{1l_1}} [F_1(s_1) e^{s_1 t_1}]$

$$= \cdots = \frac{1}{(2\pi j)^n} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} \cdots \int_{\sigma_n - j\infty}^{\sigma_n + j\infty} F(s_1, \dots, s_n) e^{s_1 t_1 + \cdots + s_n t_n} ds_1 \cdots ds_n$$

$$= \sum_{l_1=1}^{k_1} \text{Res}_{s_1=s_{1l_1}} \left\{ \cdots \left\{ \sum_{l_n=1}^{k_n} \text{Res}_{s_n=s_{nl_n}} \left\{ \sum_{l_n=1}^{k_n} \text{Res}_{s_n=s_{nl_n}} [F(s_1, \dots, s_n) e^{s_n t_n}] \right\} \cdot e^{s_{n-1} t_{n-1}} \right\} \cdots \right\} e^{s_1 t_1}.$$

证毕.

**例 2** 求函数  $F(s_1, s_2, s_3) = \frac{1}{(s_1 - a)s_2 s_3} + \frac{1}{s_1(s_2 - b)s_3} + \frac{1}{s_1 s_2(s_3 - c)}$  的三维 Laplace

变换( $a, b, c \in R$ ).

**解** 先设  $a, b, c$  均不为零.  $F(s_1, s_2, s_3)$  关于  $s_3$  只有两个一阶极点:  $s_{31} = 0$ ,  $s_{32} = c$ , 且  $\lim_{s_3 \rightarrow \infty} F(s_1, s_2, s_3) = 0$ , 故

$$F_2(s_1, s_2) = \operatorname{Res}_{s_3=0} [F(s_1, s_2, s_3)e^{s_3 t_3}] + \operatorname{Res}_{s_3=c} [F(s_1, s_2, s_3)e^{s_3 t_3}]$$

$$= \frac{1}{(s_1 - a)s_2} + \frac{1}{s_1(s_2 - b)} + \frac{1}{s_1 s_2} \cdot e^{c t_3}.$$

$F_2(s_1, s_2)$  关于  $s_2$  有两个一阶极点:  $s_{21}=0, s_{22}=b$ , 且  $\lim_{s_2 \rightarrow \infty} F_2(s_1, s_2)=0$ , 故

$$F_1(s_1) = \operatorname{Res}_{s_2=0} [F_2(s_1, s_2)e^{s_2 t_2}] + \operatorname{Res}_{s_2=b} [F_2(s_1, s_2)e^{s_2 t_2}]$$

$$= \frac{1}{s_1 - a} + \frac{1}{s_1} e^{b t_2} + \frac{1}{s_1} e^{c t_3}.$$

$F_1(s_1)$  关于  $s_1$  有两个一阶极点:  $s_{11}=0, s_{12}=a$ , 且  $\lim_{s_1 \rightarrow \infty} F_1(s_1)=0$ , 故

$$f(t_1, t_2, t_3) = \mathcal{L}^{-1}[F(s_1, s_2, s_3)] = \operatorname{Res}_{s_1=0} [F_1(s_1)e^{s_1 t_1}] + \operatorname{Res}_{s_1=a} [F_1(s_1)e^{s_1 t_1}]$$

$$= e^{a t_1} + e^{b t_2} + e^{c t_3}.$$

若  $a, b, c$  中至少有一个为零, 则结果不变.

## 2 性质

为了讨论多维 Laplace 变换的应用, 我们先介绍它的几条基本性质. 这些性质无论对于计算多维 Laplace 变换和逆变换, 还是在应用中均是很有用的. 在以下的性质中, 凡是要求多维 Laplace 变换的函数均假设满足存在定理的条件, 这些函数的增长指数统一地取为  $c_1, c_2, \dots, c_n$ . 为节省篇幅, 所有证明均略去.

### 2.1 线性性质

若  $\alpha, \beta$  是常数,  $\mathcal{L}[f_i(t_1, \dots, t_n)] = F_i(s_1, \dots, s_n), (i=1, 2)$ , 则

$$\mathcal{L}[\alpha f_1(t_1, \dots, t_n) + \beta f_2(t_1, \dots, t_n)] = \alpha \mathcal{L}[f_1(t_1, \dots, t_n)] + \beta \mathcal{L}[f_2(t_1, \dots, t_n)];$$

$$\mathcal{L}^{-1}[\alpha F_1(s_1, \dots, s_n) + \beta F_2(s_1, \dots, s_n)] = \alpha \mathcal{L}^{-1}[F_1(s_1, \dots, s_n)] + \beta \mathcal{L}^{-1}[F_2(s_1, \dots, s_n)].$$

### 2.2 微分性质

#### a. 象原函数的微分性质

若  $\mathcal{L}[f(t_1, \dots, t_n)] = F(s_1, \dots, s_n)$ , 则有

$$\mathcal{L}\left[\frac{\partial}{\partial t_{k_1}} \cdots \frac{\partial}{\partial t_{k_r}} f(t_1, \dots, t_n)\right] = \sum_{l_1, \dots, l_r=0, 1} (-1)^{l_1 + \dots + l_r} s_{k_1}^{l_1} s_{k_2}^{l_2} \cdots s_{k_r}^{l_r} \cdot \mathcal{L}[f(t_1, \dots, t_n)]|_{t_{k_1}=l_1 t_{k_1}, \dots, t_{k_r}=l_r t_{k_r}}^*$$

$$(1 \leq k_1 < k_2 < \cdots < k_r \leq n, r = n - i + l_1 + \cdots + l_r, i = 1, 2, \dots, n) \quad (2)$$

\* 这里  $\mathcal{L}$  表示在  $t_1, t_2, \dots, t_n$  中对于剩余的  $r$  个非零变量取  $r$  维 Laplace 变换; 当  $t_m=0$  时, 相应的函数值是指右极限值. 下同.

$$\mathcal{L}\left[\left(\frac{\partial}{\partial t_i}\right)^k f(t_1, \dots, t_n)\right] = s_i^k F(s_1, \dots, s_n) - \sum_{l=0}^{k-1} s_i^{k-1-l} \cdot \mathcal{L}_{n-1}\left[\left(\frac{\partial}{\partial t_i}\right)^l f(t_1, \dots, t_n)\right]_{t_i=0} \quad (i=1, \dots, n, k \in N) \quad (3)$$

注: 利用以上公式逐步递推, 可以求得

$$\mathcal{L}\left[\left(\frac{\partial}{\partial t_1}\right)^{k_1} \left(\frac{\partial}{\partial t_2}\right)^{k_2} \cdots \left(\frac{\partial}{\partial t_n}\right)^{k_n} f(t_1, t_2, \dots, t_n)\right] \quad (k_1, k_2, \dots, k_n = 0, 1, 2, \dots).$$

### b. 象函数的微分性质

若  $\mathcal{L}[f(t_1, \dots, t_n)] = F(s_1, \dots, s_n)$ , 则有

$$\left(\frac{\partial}{\partial s_1}\right)^{k_1} \left(\frac{\partial}{\partial s_2}\right)^{k_2} \cdots \left(\frac{\partial}{\partial s_n}\right)^{k_n} F(s_1, s_2, \dots, s_n) = \mathcal{L}[(-t_1)^{k_1} (-t_2)^{k_2} \cdots (-t_n)^{k_n} f(t_1, \dots, t_n)],$$

$$(k_1, k_2, \dots, k_n = 0, 1, 2, \dots, \operatorname{Re}(s_i) > c_i, i = 1, 2, \dots, n).$$

## 2.3 积分性质

### a. 象原函数的积分性质

若  $\mathcal{L}[f(t_1, \dots, t_n)] = F(s_1, \dots, s_n)$ , 则有

$$\mathcal{L}\left\{\underbrace{\int_0^{t_1} dt_1 \cdots \int_0^{t_n} dt_n}_{r_1 \text{ 次}} \cdots \underbrace{\left\{\int_0^{t_1} dt_1 \cdots \int_0^{t_n} f(t_1, \dots, t_n) dt_1 \cdots dt_n\right\}}_{r_1 \text{ 次}} \cdots \right\} = \frac{F(s_1, s_2, \dots, s_n)}{s_1^{r_1} s_2^{r_2} \cdots s_n^{r_n}},$$

$$(r_i = 0, 1, 2, \dots, \operatorname{Re}(s_i) > \max\{0, c_i\}, i = 1, \dots, n).$$

### b. 象函数的积分性质

若  $\mathcal{L}[f(t_1, \dots, t_n)] = F(s_1, \dots, s_n)$ , 则有

$$\mathcal{L}\left[\frac{f(t_1, \dots, t_n)}{t_1^{r_1} t_2^{r_2} \cdots t_n^{r_n}}\right] = \underbrace{\int_{s_1}^{\infty} ds_1 \cdots \int_{s_n}^{\infty} ds_n}_{r_n \text{ 次}} \left\{ \underbrace{\int_{s_1}^{\infty} ds_1 \cdots \int_{s_1}^{\infty} F(s_1, \dots, s_n) ds_1}_{r_1 \text{ 次}} \cdots \right\},$$

$$(r_i = 0, 1, 2, \dots, \operatorname{Re}(s_i) > c_i, i = 1, 2, \dots, n).$$

## 2.4 分离性质

若  $f(t_1, \dots, t_n) = f_1(t_{k_0+1}, \dots, t_{k_1}) f_2(t_{k_1+1}, \dots, t_{k_2}) \cdots f_r(t_{k_{r-1}+1}, \dots, t_k)$ ,  $\mathcal{L}[f(t_1, \dots, t_n)] = F(s_1, \dots, s_n)$ ,

$$F_i(s_{k_{i-1}+1}, \dots, s_{k_i}) = \mathcal{L}_{k_i-k_{i-1}}[f_i(t_{k_{i-1}+1}, \dots, t_{k_i})] = \int_0^{+\infty} \cdots \int_0^{+\infty} f_i(t_{k_{i-1}+1}, \dots, t_{k_i}) e^{-(s_{k_{i-1}+1} t_{k_{i-1}+1} + \cdots + s_{k_i} t_{k_i})}$$

$$dt_{k_{i-1}+1} \cdots dt_{k_i}, \quad (0 = k_0 < k_1 < \cdots < k_r = n, i = 1, 2, \dots, r),$$

$$\text{则 } F(s_1, \dots, s_n) = F_1(s_{k_0+1}, \dots, s_{k_1}) F_2(s_{k_1+1}, \dots, s_{k_2}) \cdots F_r(s_{k_{r-1}+1}, \dots, s_k).$$

## 2.5 位移性质

若  $a_1, a_2, \dots, a_n$  为常数,  $\mathcal{L}[f(t_1, \dots, t_n)] = F(s_1, \dots, s_n)$ , 则有

$$\mathcal{L}[e^{a_1 t_1 + \dots + a_n t_n} \cdot f(t_1, \dots, t_n)] = F(s_1 - a_1, \dots, s_n - a_n), \quad (\operatorname{Re}(s_i - a_i) > c_i, i = 1, 2, \dots, n).$$

## 2.6 延迟性质

若  $\mathcal{L}[f(t_1, \dots, t_n)] = F(s_1, \dots, s_n)$ , 当  $t_i < 0 (i = 1, \dots, n)$  时,  $f(t_1, \dots, t_n) = 0$ , 则对于任意  $\tau_1 \geq 0, \tau_2 \geq 0, \dots, \tau_n \geq 0$ , 有

$$\mathcal{L}[f(t_1 - \tau_1, t_2 - \tau_2, \dots, t_n - \tau_n)] = e^{-(s_1 \tau_1 + \dots + s_n \tau_n)} F(s_1, \dots, s_n).$$

## 2.7 相似性质

若  $\mathcal{L}[f(t_1, \dots, t_n)] = F(s_1, \dots, s_n)$ ,  $a_1 > 0, a_2 > 0, \dots, a_n > 0$ , 则

$$\mathcal{L}[f(a_1 t_1, a_2 t_2, \dots, a_n t_n)] = \frac{1}{a_1 a_2 \dots a_n} F\left(\frac{s_1}{a_1}, \frac{s_2}{a_2}, \dots, \frac{s_n}{a_n}\right), \quad (\operatorname{Re}(s_i) > a_i c_i, i = 1, 2, \dots, n).$$

## 2.8 乘积性质

设  $f_1(t_1, \dots, t_n)$  和  $f_2(t_1, \dots, t_n)$  的增长指数分别为  $c_1, \dots, c_n$  和  $d_1, \dots, d_n$ ,  $\mathcal{L}[f_i(t_1, \dots, t_n)] = F_i(s_1, \dots, s_n)$ ,  $(i = 1, 2)$ , 则

$$\begin{aligned} \mathcal{L}[f_1(t_1, \dots, t_n) f_2(t_1, \dots, t_n)] &= \frac{1}{(2\pi j)^n} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} \dots \int_{\sigma_n - j\infty}^{\sigma_n + j\infty} F_1(p_1, p_2, \dots, p_n) \\ &\quad \cdot F_2(s_1 - p_1, s_2 - p_2, \dots, s_n - p_n) dp_1 \dots dp_n, \quad (\sigma_i > c_i, \operatorname{Re}(p_i) = \sigma_i, \operatorname{Re}(s_i) > \sigma_i + d_i, i = 1, 2, \dots, n). \end{aligned}$$

## 2.9 卷积性质

如果  $f_1(t_1, \dots, t_n)$  和  $f_2(t_1, \dots, t_n)$  满足存在定理条件, 则它们的卷积定义为

$$f_1(t_1, \dots, t_n) * f_2(t_1, \dots, t_n) = \int_0^{t_1} \int_0^{t_2} \dots \int_0^{t_n} f_1(\tau_1, \tau_2, \dots, \tau_n) f_2(t_1 - \tau_1, t_2 - \tau_2, \dots, t_n - \tau_n) d\tau_1 \dots d\tau_n.$$

我们有如下的卷积性质:

若  $\mathcal{L}[f_i(t_1, \dots, t_n)] = F_i(s_1, \dots, s_n)$ ,  $i = 1, 2$ ,

则有  $\mathcal{L}[f_1(t_1, \dots, t_n) * f_2(t_1, \dots, t_n)] = F_1(s_1, \dots, s_n) F_2(s_1, \dots, s_n)$ .

**例3** 设  $f(t_1, t_2, t_3, t_4) = t_3 e^{2t_4} \cdot \cos(t_1 + t_2)$ , 求  $\mathcal{L}[f(t_1, t_2, t_3, t_4)]$ .

**解** 由分离性质有

$$F(s_1, s_2, s_3, s_4) = \mathcal{L}[f(t_1, t_2, t_3, t_4)] = \mathcal{L}_1[t_3] \cdot \mathcal{L}_1[e^{2t_4}] \cdot \mathcal{L}_2[\cos(t_1 + t_2)].$$

而  $\mathcal{L}_1[t_3] = \frac{1}{s_3^2}$ ,  $\mathcal{L}_1[e^{2t_4}] = \frac{1}{s_4 - 2}$ .

对于  $\mathcal{L}_2[\cos(t_1 + t_2)]$ , 由于  $\frac{\partial^2}{\partial t_1 \partial t_2} \cos(t_1 + t_2) = -\cos(t_1 + t_2)$ , 根据微分性质有:

$$\begin{aligned} -\mathcal{L}_2[\cos(t_1 + t_2)] &= \mathcal{L}_2\left[\frac{\partial^2}{\partial t_1 \partial t_2} \cos(t_1 + t_2)\right] \\ &= s_1 s_2 \mathcal{L}_2[\cos(t_1 + t_2)] - s_1 \mathcal{L}_1[\cos t_1] - s_2 \mathcal{L}_1[\cos t_2] + 1 \\ &= s_1 s_2 \mathcal{L}_2[\cos(t_1 + t_2)] - \frac{s_1^2}{s_1^2 + 1} - \frac{s_2^2}{s_2^2 + 1} + 1 \\ &= s_1 s_2 \mathcal{L}_2[\cos(t_1 + t_2)] - \frac{s_1^2 s_2^2 - 1}{(s_1^2 + 1)(s_2^2 + 1)}, \end{aligned}$$

故  $\mathcal{L}_2[\cos(t_1 + t_2)] = \frac{s_1 s_2 - 1}{(s_1^2 + 1)(s_2^2 + 1)}$ .

这样,  $F(s_1, s_2, s_3, s_4) = \frac{s_1 s_2 - 1}{(s_1^2 + 1)(s_2^2 + 1)s_3^2(s_4 - 2)}$ .

### 3 应用

多维 Laplace 变换可用来解偏微分方程(组), 其方法和步骤跟一维时相类似. 首先, 在偏微分方程(组)两边对某些变量取多维 Laplace 变换就能消去未知函数对这些自变量求偏导数的运算, 从而得到象函数的较为简单的偏微分方程(组)或代数方程(组); 其次, 再解这个方程(组)而得象函数; 最后, 对象函数取多维 Laplace 逆变换便可得象原函数即为所求偏微分方程(组)的解. 从下面几例可看出, 一些用现有方法难于求解的方程, 用多维 Laplace 变换均能简便地求解. 这就说明, 在应用中多维 Laplace 变换比一维 Laplace 变换具有更大的广泛性和优越性.

#### 例 4 解偏微分方程组

$$\begin{cases} \frac{\partial^2 u_1}{\partial t_1 \partial t_2} - 2 \frac{\partial u_1}{\partial t_1} + \frac{\partial u_2}{\partial t_2} + u_2 = t_1 + t_2 + 1, \\ 2 \frac{\partial^2 u_2}{\partial t_1 \partial t_2} - \frac{\partial u_2}{\partial t_1} + \frac{\partial u_1}{\partial t_1} + u_1 = (t_1 + 1)e^{2t_2} - 1, & -\infty < t_1, t_2 < +\infty, \\ u_1|_{t_1=0} = 0, u_1|_{t_2=0} = t_1, u_2|_{t_1=0} = t_2, u_2|_{t_2=0} = t_1. \end{cases}$$

**解** 记  $U_i(s_1, s_2) = \mathcal{L}[u_i(t_1, t_2)]$ , ( $i = 1, 2$ ). 利用微分性质和定解条件在方程组两边对  $t_1, t_2$  取二维 Laplace 变换可得:

$$\begin{cases} s_1 s_2 U_1 - \frac{1}{s_1} - 2s_1 U_1 + s_2 U_2 - \frac{1}{s_1^2} + U_2 = \frac{1}{s_1^2} + \frac{1}{s_1 s_2^2} + \frac{1}{s_1 s_2}, \\ 2s_1 s_2 U_2 - \frac{2}{s_1} - \frac{2}{s_2} - s_1 U_2 + \frac{1}{s_2^2} + s_1 U_1 + U_1 = \frac{s_1 + 1}{s_1^2(s_2 - 2)} - \frac{1}{s_1 s_2}. \end{cases}$$

解此代数方程组得

$$\begin{cases} U_1(s_1, s_2) = \frac{1}{s_1^2(s_2 - 1)}, \\ U_2(s_1, s_2) = \frac{s_1 + s_2}{s_1^2 s_2^2}. \end{cases}$$

取二维 Laplace 逆变换可得所求偏微分方程组之解为

$$\begin{cases} u_1(t_1, t_2) = t_1 e^{2t_1}, \\ u_2(t_1, t_2) = t_1 + t_2. \end{cases}$$

**例5** 解第一卦限内的三维 Laplace 方程的边值问题

$$\begin{cases} \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, & x > 0, y > 0, z > 0, \\ u|_{x=0} = f_1(y, z), u|_{y=0} = f_2(x, z), u|_{z=0} = f_3(x, y), \\ \frac{\partial u}{\partial x} \Big|_{x=0} = f_4(y, z), \frac{\partial u}{\partial y} \Big|_{y=0} = f_5(x, z), \frac{\partial u}{\partial z} \Big|_{z=0} = f_6(x, y). \end{cases}$$

**解** 记  $U(s_1, s_2, s_3) = \mathcal{L}[u(x, y, z)]$ ,  $F_1(s_2, s_3) = \mathcal{L}_2[f_1(y, z)]$ ,  $F_2(s_1, s_3) = \mathcal{L}_2[f_2(x, z)]$ ,  $F_3(s_1, s_2) = \mathcal{L}_2[f_3(x, y)]$ ,  $F_4(s_2, s_3) = \mathcal{L}_2[f_4(y, z)]$ ,  $F_5(s_1, s_3) = \mathcal{L}_2[f_5(x, z)]$ ,  $F_6(s_1, s_2) = \mathcal{L}_2[f_6(x, y)]$ . 利用微分性质和边界条件在方程两边取三维 Laplace 变换可得

$$s_1^2 U - s_1 F_1 - F_4 + s_2^2 U - s_2 F_2 - F_5 + s_3^2 U - s_3 F_3 - F_6 = 0.$$

解得 
$$U(s_1, s_2, s_3) = \frac{1}{s_1^2 + s_2^2 + s_3^2} [s_1 F_1 + s_2 F_2 + s_3 F_3 + F_4 + F_5 + F_6].$$

取三维 Laplace 逆变换便得所求边值问题的解为

$$u = u(x, y, z) = \mathcal{L}^{-1} \left\{ \frac{1}{s_1^2 + s_2^2 + s_3^2} [s_1 F_1 + s_2 F_2 + s_3 F_3 + F_4 + F_5 + F_6] \right\}.$$

特别地, 若取  $f_1(y, z) = -2y^2 + 2z^2$ ,  $f_2(x, z) = x + 2z^2$ ,  $f_3(x, y) = x - 2y^2$ ,  $f_4(y, z) = 1$ ,  $f_5(x, z) = f_6(x, y) = 0$ , 则有  $F_1(s_2, s_3) = \frac{-4}{s_2^2 s_3} + \frac{4}{s_2 s_3^3}$ ,  $F_2(s_1, s_3) = \frac{1}{s_1^2 s_3} + \frac{4}{s_1 s_3^3}$ ,  $F_3(s_1, s_2) = \frac{1}{s_1^2 s_2} - \frac{4}{s_1 s_2^3}$ ,  $F_4(s_2, s_3) = \frac{1}{s_2 s_3}$ ,  $F_5(s_1, s_3) = F_6(s_1, s_2) = 0$ , 所以

$$U(s_1, s_2, s_3) = \frac{1}{s_1^2 s_2 s_3} - \frac{4}{s_1 s_2^3 s_3} + \frac{4}{s_1 s_2 s_3^3},$$



故  $u = u(x, y, z) = \mathcal{L}^{-1}[U(s_1, s_2, s_3)] = x - 2y^2 + 2z^2$ .

### 例6 解偏微分方程

$$\begin{cases} \frac{\partial^3 u}{\partial t_1^3} + 2 \frac{\partial^2 u}{\partial t_2 \partial t_3} - \frac{\partial u}{\partial t_4} = 4t_1 t_2 - e^{t_4}, & -\infty < t_1, t_2, t_3, t_4 < +\infty, \\ u|_{t_1=0} = u|_{t_2=0} = u|_{t_3=0} = e^{t_4}, \quad u|_{t_4=0} = 1 + t_1 t_2^2 t_3, \\ \frac{\partial u}{\partial t_1} \Big|_{t_1=0} = t_2^2 t_3, \quad \frac{\partial^2 u}{\partial t_1^2} \Big|_{t_1=0} = 0. \end{cases}$$

解 记  $U(s_1, s_2, s_3, s_4) = \mathcal{L}[u(t_1, t_2, t_3, t_4)]$ . 利用微分性质和定解条件在方程两边取四维 Laplace 变换可得

$$\begin{aligned} s_1^3 U - \frac{s_1^2}{s_2 s_3 (s_4 - 1)} - \frac{2s_1}{s_2^3 s_3 s_4} + 2s_2 s_3 U - \frac{2s_2}{s_1 s_2 (s_4 - 1)} - \frac{2s_3}{s_1 s_3 (s_4 - 1)} + \frac{2}{s_1 (s_4 - 1)} \\ - s_4 U + \frac{1}{s_1 s_2 s_3} + \frac{2}{s_1^2 s_2^2 s_3^2} = \frac{4}{s_1^2 s_2^2 s_3 s_4} - \frac{1}{s_1 s_2 s_3 (s_4 - 1)}. \end{aligned}$$

解得 
$$U(s_1, s_2, s_3, s_4) = \frac{2}{s_1^2 s_2^2 s_3 s_4} + \frac{1}{s_1 s_2 s_3 (s_4 - 1)}.$$

取四维 Laplace 逆变换可得所求偏微分方程之解为

$$u(t_1, t_2, t_3, t_4) = t_1 t_2^2 t_3 + e^{t_4}.$$

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## Multidimensional Laplace transform and its application

Fan Hongfu

(Department of Basic Studies, ECUT)

**Abstract** In this paper, the notion of multidimensional Laplace transform is introduced, and its application to solve differential equations is discussed.

**Keywords** Laplace transform; differential equation; residue